

# The Non-Negativity of Probabilities and the Collapse of State

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The dynamical equation, being the combination of Schrödinger and Liouville equations, produces noncausal evolution when the initial state of interacting quantum and classical mechanical systems is as it is demanded in discussions regarding the problem of measurement. It is found that state of quantum mechanical system instantaneously collapses due to the non-negativity of probabilities.

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## 1. INTRODUCTION

Quantum and classical mechanics are causal theories. By this we mean that during evolutions, that are governed by dynamical equations of these theories, states cannot change their purities. (Of course, this holds only in the cases with no stochastic terms in the Hamiltonian.) However, there are situations in which purity of state can be changed. This noncontinuous change happens when quantum system interacts with some classical system. An example of this is a process of measurement with a well-known reduction, collapse, of state.

Theory that unifies quantum and classical mechanics by describing interaction of classical and quantum systems has to be based on such dynamical equation which can produce noncausal evolutions. The dynamical equation of hybrid systems, quantum and classical systems in interaction, which was firstly introduced by Aleksandrov (1981), produces noncausal evolution in a case addressing the problem of measurement. More precisely, If the state of quantum system before the measurement is the superposition of the eigenstates of measured observable, i.e.,  $\sum_i |\psi_i\rangle$ , and if the apparatus before the measurement is in the state with sharp values of position and momentum, then the pure initial noncorrelated state has to evolve into some mixed correlated state. The equation of motion governing this process is just the combination of Schrödinger–von Neumann, and Liouville equations. Interesting is that for this transition only the regular type

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Hamiltonian is needed (that is Hamiltonian with no stochastic terms). On the other hand, important role is played by the non-negativity of states, which resembles the non-negativity of probabilities, and this is what we shall discuss in this article.

In order to investigate mentioned non-negativity of states, we shall introduce operator form of classical mechanics. Our approach to this problem is very similar to the one proposed by Sudarshan and coworkers (Gautam *et al.*, 1979; Shery and Sudarshan, 1978, 1979).

## 2. OPERATOR FORM OF CLASSICAL MECHANICS

The classical mechanics, in difference to quantum mechanics, is characterized by the possibility of simultaneous measurement of both position and momentum with vanishing deviations. Due to this, the algebra representing observables of classical mechanics has to be the commutative one. In the direct product of two rigged Hilbert spaces  $\mathcal{H}^q \otimes \mathcal{H}^p$  one can define commutative algebra of classical observables as the algebra (over  $\mathbf{R}$ ) of polynomials of the operators  $\hat{q}_{\text{cm}} = \hat{q} \otimes \hat{I}$  and  $\hat{p}_{\text{cm}} = \hat{I} \otimes \hat{p}$ . These operators represent coordinate and momentum of classical system. States can be defined, like in standard phase space formulation, as functions of position and momentum, which are now operators. That is, pure states are defined by:

$$\begin{aligned} \delta(\hat{q} - q(t)) \otimes \delta(\hat{p} - p(t)) &= \int \int \delta(q - q(t)) \delta(p - p(t)) |q\rangle \langle q| \otimes |p\rangle \langle p| dq dp \\ &= |q(t)\rangle \langle q(t)| \otimes |p(t)\rangle \langle p(t)|, \end{aligned} \quad (1)$$

while (noncoherent) mixtures are  $\rho(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}, t)$ . These states are positive and Hermitian operators normalized to  $\delta^2(0)$  if  $\rho(q, p, t) \in \mathbf{R}$ ,  $\rho(q, p, t) \geq 0$  and  $\int \int \rho(q, p, t) dq dp = 1$ . If one calculates the mean values by the Ansatz:

$$\langle f \rangle = \frac{\text{Tr} f(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}) \rho(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}, t)}{\text{Tr} \rho(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}})}, \quad (2)$$

then  $\langle f \rangle$  will be equal to standardly calculated:

$$\bar{f} = \int \int f(q, p) \rho(q, p, t) dq dp. \quad (3)$$

The dynamical equation in operator formulation is defined as:

$$\begin{aligned} \frac{\partial \rho(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}, t)}{\partial t} &= \frac{\partial H(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}})}{\partial \hat{q}_{\text{cm}}} \frac{\partial \rho(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}, t)}{\partial \hat{p}_{\text{cm}}} \\ &\quad - \frac{\partial \rho(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}, t)}{\partial \hat{q}_{\text{cm}}} \frac{\partial H(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}})}{\partial \hat{p}_{\text{cm}}}. \end{aligned} \quad (4)$$

It is obvious that this form is equivalent to the standard classical mechanics since the latter appears through the kernels of the operator formulation (expressed with respect to the basis  $|q\rangle \otimes |p\rangle$ ). Let us further remark that this formulation of classical mechanics employs formalism of standard quantum mechanics. More precisely, the direct product of two rigged Hilbert spaces  $\mathcal{H}^q \otimes \mathcal{H}^p$  used here is “the carbon copy” of the one used in quantum mechanics when the coordinates of system with two degrees of freedom are under consideration. The only difference comes from the fact that we have neglected non-commuting operators here since they have no physical meaning. All other aspects of the formalism are the same or similar. Without going into details since it is beyond the scope of this article, it should be stressed that this holds for all formal problems and respective solutions as well.

### 3. HYBRID SYSTEMS

One can use operator form of classical mechanics in order to analyze the interaction between classical and quantum systems. Mathematical framework is based on direct product of the Hilbert space and two rigged Hilbert spaces (in case when considered classical and quantum systems have only one degree of freedom). The first Hilbert space  $\mathcal{H}_{\text{qm}}$  is as in the standard quantum mechanics, while the other two are rigged Hilbert spaces that were discussed in previous section. So, for description of so called hybrid systems one uses  $\mathcal{H}_{\text{qm}} \otimes \mathcal{H}_{\text{cm}}^q \otimes \mathcal{H}_{\text{cm}}^p$ .

The state of the composite system is the statistical operator  $\hat{\rho}_{\text{qm}}(t) \otimes \hat{\rho}_{\text{cm}}(t)$ , where the first one acts in  $\mathcal{H}_{\text{qm}}$  representing the state of quantum system and second one acts in  $\mathcal{H}_{\text{cm}}^q \otimes \mathcal{H}_{\text{cm}}^p$  representing the classical system. The properties of these operators are as in standard quantum mechanics and as given in previous section.

The evolution of hybrid systems state is governed by the Hamiltonian  $\hat{H} = \hat{V}_{\text{qm}} \otimes \hat{V}_{\text{cm}}$ , where

$$\hat{V}_{\text{qm}} = V_{\text{qm}}(\hat{q}_{\text{qm}}, \hat{p}_{\text{qm}}) = V_{\text{qm}}(\hat{q} \otimes \hat{I} \otimes \hat{I}, \hat{p} \otimes \hat{I} \otimes \hat{I}),$$

and

$$\hat{V}_{\text{cm}} = V_{\text{cm}}(\hat{q}_{\text{cm}}, \hat{p}_{\text{cm}}) = V_{\text{cm}}(\hat{I} \otimes \hat{q} \otimes \hat{I}, \hat{I} \otimes \hat{I} \otimes \hat{p}).$$

Since it is Hermitian, operator  $\hat{V}_{\text{qm}}$  can be diagonalized in form:

$$\sum_i v_i |\psi_i\rangle \langle \psi_i| \otimes \hat{I} \otimes \hat{I}.$$

Obviously, the operator  $\hat{V}_{\text{cm}}$  is diagonal with respect to the basis  $|q\rangle \otimes |p\rangle$ .

The dynamical equation for hybrid systems is the generalization of Schrödinger and Liouville equations or, more precisely, their combination given

by:

$$\begin{aligned} \frac{\partial \hat{\rho}_{\text{qm}}(t) \otimes \hat{\rho}_{\text{cm}}(t)}{\partial t} = & \frac{1}{i\hbar} [\hat{V}_{\text{qm}}, \hat{\rho}_{\text{qm}}(t)] \otimes \hat{\rho}_{\text{cm}}(t) \hat{V}_{\text{cm}} \\ & + \frac{\hat{\rho}_{\text{qm}}(t) \hat{V}_{\text{qm}} + \hat{V}_{\text{qm}} \hat{\rho}_{\text{qm}}(t)}{2} \otimes \{\hat{V}_{\text{cm}}, \hat{\rho}_{\text{cm}}(t)\}, \end{aligned} \quad (5)$$

where operator form of the Poisson bracket  $\{, \}$  is defined by (4). Similar equation appeared in (Aleksandrov, 1981, 1995; Boucher and Trashen, 1988; Prezhdo and Kisil, 1997). There one can find detailed discussions regarding the properties of there given dynamical equations of hybrid systems.

#### 4. PROCESS OF MEASUREMENT

In literature, an ideal quantum measurement is considered as interaction between the quantum system, described by the state  $|\psi(t)\rangle$  in a Hilbert space  $\mathcal{H}_{\text{qm}}$ , and the measuring apparatus, classical system, initially in the state  $|\phi(t_0)\rangle$ . The measurement process is such that (a) the quantum system, before the measurement being in one of the eigenstates of the measured observable, say  $|\psi_i(t_0)\rangle$ , does not change the state during the measurement (repeated measurement has to give the same results) and (b) the classical system undergoes transition from initial state  $|\phi(t_0)\rangle$  to  $|\phi_i(t)\rangle$ . This transition enables one to find out what is the state of measured quantum mechanical system.

The problem of the measurement is the following: if the initial state of the quantum system was superposition of the eigenstates of measured observable, that is if  $|\psi(t_0)\rangle = \sum_i c_i |\psi_i(t_0)\rangle$ , then, due to assumed linearity of the evolution, the state of the composite system would be  $|\psi(t)\rangle = \sum_i c_i |\psi_i(t)\rangle \otimes |\phi_i(t)\rangle$ , which is in contradiction with the obvious fact that classical system cannot be in superposed states. Many other processes can be related to this one in more or less straightforward manner.

Within the operator formulation of the classical mechanics and hybrid systems, the process of measurement can be described as follows. The initial state:

$$\hat{\rho}_{\text{qm}}(t_0) \otimes \hat{\rho}_{\text{cm}}(t_0) = \sum_i \sum_j c_i c_j^* |\psi_i(t_0)\rangle \langle \psi_j(t_0)| \otimes |q(t_0)\rangle \langle q(t_0)| \otimes |p(t_0)\rangle \langle p(t_0)|, \quad (6)$$

evolves according to the dynamical equation (5) where  $\hat{V}_{\text{qm}}$  is the measured observable. The last term on the RHS of (5), due to which  $\hat{\rho}_{\text{cm}}(t)$  depends on  $\hat{\rho}_{\text{qm}}(t)$ , in this case becomes:

$$\sum_i \sum_j \frac{v_i + v_j}{2} c_i c_j^* |\psi_i(t)\rangle \langle \psi_j(t)| \otimes \{\hat{V}_{\text{cm}}, \hat{\rho}_{\text{cm}}(t)\}.$$

This term suggests that correlated state can be assumed in the form of:

$$\sum_i \sum_j c_i c_j^* |\psi_i(t)\rangle \langle \psi_j(t)| \otimes |q_{ij}(t)\rangle \langle q_{ij}(t)| \otimes |p_{ij}(t)\rangle \langle p_{ij}(t)|. \tag{7}$$

But, such operator, despite of being the solution of (5), is not non-negative one, i.e., some events would have negative probabilities if this operator is taken as the state of composite system. The only meaningful solution of dynamical equation is:

$$\sum_i |c_i|^2 |\psi_i\rangle \langle \psi_i| \otimes |q_i(t)\rangle \langle q_i(t)| \otimes |p_i(t)\rangle \langle p_i(t)|. \tag{8}$$

This operator is Hermitian and positive one.

### 5. DISCUSSION

The initial state of hybrid systems (6) is idempotent (up to the norm), while the evolved state in considered case (8) is not. Thus, in the absence of some *ad hoc* introduced stochastic terms in the Hamiltonian and/or nonlinear terms in the equation of motion, this equation produces noncausal evolution: the initial noncorrelated pure state evolves in mixed correlated state.

From the evolved state it follows that to each state of the measured quantum system  $|\psi_i\rangle$  (which is the eigenstate of the measured observable), there corresponds one state of the measuring apparatus (with sharp values of position and momentum)  $|q_i(t)\rangle \otimes |p_i(t)\rangle$  and each of these states happens with the probability  $|c_i(t_0)|^2$ . Consequently, solution (8) is in agreement with the projection postulate of orthodox quantum mechanics.

The formal description of the collapse of quantum mechanical state could be the following. Initial state of the hybrid system should be seen as

$$\sum_{ij} c_{ij}(t) |\psi_i\rangle \langle \psi_j| \otimes |q_i(t)\rangle \langle q_j(t)| \otimes |p_i(t)\rangle \langle p_j(t)|$$

for  $t = t_0$  since this correlated state is designed to be as pure, Hermitian and non-negative for  $t \geq t_0$  as is the initial one. The partial derivations within the Poisson bracket on the right-hand-side of the dynamical equation, which is the generator of time transformation, for  $t > t_0$  annihilate nondiagonal classical mechanical terms of the state according to

$$\frac{\partial}{\partial \hat{q}} |q_i(t)\rangle \langle q_j(t)| = \frac{\partial}{\partial \hat{q}} \delta(\hat{q} - q_i(t)) \delta_{i,j},$$

$$\frac{\partial}{\partial \hat{p}} |p_i(t)\rangle \langle p_j(t)| = \frac{\partial}{\partial \hat{p}} \delta(\hat{p} - p_i(t)) \delta_{i,j},$$

since the classical mechanical  $i \neq j$  terms of designed state for  $t > t_0$  do not commute with coordinate and momentum of the classical system, the meaning of which is that they are not functions of the only available observables. For  $t = t_0$  these derivatives do not vanish since  $q_i(t_0) = q_0$  and  $p_i(t_0) = p_0$  for all  $i$ . This means that dynamical equation instantaneously changes  $i \neq j$  terms of classical mechanical state at  $t_0$  and then forbids further time evolution of these terms, i.e., these terms become constant. Since there is no other possibility for the state of hybrid system to be non-negative operator,  $i \neq j$  terms of classical mechanical state has to vanish in order to be time independent and, in this way, they annihilate  $i \neq j$  terms of quantum mechanical state. This is seen as the collapse of state of quantum mechanical system.

Similar reasoning holds in some other cases of the interaction between classical and quantum systems. The pure initial states can evolve in noncoherent mixtures, while noncoherent mixtures cannot evolve into coherent mixtures (pure states), i.e. the process is irreversible. Therefore, the entropy increases or stays constant as the consequence of the superposition of two linear dynamical equations and the non-negativity of probability.

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